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Linearly anisotropic scattering in a rectangular medium exposed to collimated radiation

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1. INTRODUCTION

Over the past three decades many studies on radiative heat transfer in anisotropically scattering media have been conducted. This is because the radiative contribution can be significant in the problem of energy transport in coal-fired furnaces, metalized propellant plumes, and particulate clouds. Most of these studies considered one-dimensional geometry [1]. In recent years, exact integral formulation has been developed for multi-dimensional anisotropic scattering. Crosbie and Dougherty [2] modeled the scattering of a laser beam in a radially infinite cylindrical medium. Lin and Tsai [3] presented integral formulation in terms of source function. Integral equations of moments of intensity for anisotropic scattering in a medium with Fresnel boundaries have been developed recently by Wu [4]. However, these authors have not reported accurate solutions of the exact integral formulation for anisotropic scattering in a two-dimensional rectangular medium whereas various solutions of the exact integral formulation for relatively simpler isotropic scattering in the same geometry have been presented [5, 6].

The purpose of this work is to present an accurate solution of the integral equations describing conservative anisotropic scattering in a two-dimensional rectangular medium exposed to collimated radiation. For anisotropic scattering, the formulation in terms of moments of intensity involves spatial variables only [4]. Since the reduction in independent variables offers a significant simplification when we develop a solution, the integral equations of moments of intensity are adopted in this work.

2. INTEGRAL FORMULATION

Define the optical coordinates as (x, y, z), which are the products of geometric coordinates and the extinction coefficient. The medium considered is a rectangular bar bounded by $y = \pm b$ and $z = 0, c$, but unbounded in the $\pm x$-direction. The origin of the coordinates is at the center of the bottom. The assumptions about the system are: (i) the medium is homogeneous, (ii) the medium is in local thermodynamic equilibrium, (iii) steady state is achieved, (iv) scattering in the medium is conservative and linearly anisotropic, (v) the index of refraction is unity, (vi) the medium does not reflect or reradiate at the boundaries, and (vii) a normal uniform collimated radiation, I_0, is incident at the bottom.

Define the source function as

$$S(s, \mathbf{\Omega}) = (1/4\pi) \int_{4\pi} I(s, \mathbf{\Omega}')(1 + a_1 \mathbf{\Omega} \cdot \mathbf{\Omega}')\, d\Omega' \tag{1}$$

where I is the radiation intensity, s an arbitrary path from the boundary to a location in the medium, $\mathbf{\Omega}$ the direction determined by the polar angle θ and the azimuthal angle ϕ, and a_1 the coefficient of anisotropic scattering. Following the procedure described in our previous work [4], one can recast the source function as

$$S(y, z, \theta, \phi) = \frac{I_0}{4\pi}[J(y, z) + a_1 \cos\theta Q_z(y, z) + a_1 \sin\theta \sin\phi Q_y(y, z)] \tag{2}$$

where J, Q_y and Q_z are dimensionless moments of the intensity. These moments satisfy the integral equations

$$J(y, z) = e^{-z} + \frac{1}{4}\int_{-b}^{b}\int_0^c \left[\frac{S_1(\tau)}{\tau} J(y', z') + a_1 \frac{S_2(\tau)}{\tau^2}(z - z')Q_z(y', z') + a_1 \frac{S_2(\tau)}{\tau^2}(y - y')Q_y(y', z')\right] dz'\, dy' \tag{3}$$

$$Q_z(y, z) = e^{-z} + \frac{1}{4}\int_{-b}^{b}\int_0^c \left[\frac{S_2(\tau)}{\tau^2}(z - z')J(y', z') + a_1 \frac{S_3(\tau)}{\tau^3}(z - z')^2 Q_z(y', z') + a_1 \frac{S_3(\tau)}{\tau^3}(y - y')(z - z')Q_y(y', z')\right] dz'\, dy' \tag{4}$$

$$Q_y(y, z) = \frac{1}{4}\int_{-b}^{b}\int_0^c \left[\frac{S_2(\tau)}{\tau^2}(y - y')J(y', z') + a_1 \frac{S_3(\tau)}{\tau^3}(y - y')(z - z')Q_z(y', z') + a_1 \frac{S_3(\tau)}{\tau^3}(y - y')^2 Q_y(y', z')\right] dz'\, dy' \tag{5}$$

where

$$\tau = [(y - y')^2 + (z - z')^2]^{1/2} \tag{6}$$

and S_n is a generalized exponential integral function defined by

$$\frac{\pi}{2} S_n(\tau) = \int_1^\infty \frac{e^{-\tau t}}{t^n (t^2 - 1)^{1/2}}\, dt. \tag{7}$$

The integral on the right-hand side of equation (7) is the so-called Bickley–Naylor function [7, 8]. Physically, J is the total radiation intensity, Q_y the y component of radiative flux and Q_z the z component of radiative flux.

3. METHOD OF SOLUTION

A simple collocation method is now applied to develop solutions to equations (3)–(5). Because of the symmetry of the radiation field in the y direction, we assume $J(y, z)$, $Q_z(y, z)$ and $Q_y(y, z)$ to be

$$J(y, z) = \sum_{m=0}^{M} \sum_{n=0}^{N} A_{mn} y^{2m} z^n \tag{8}$$

$$Q_z(y, z) = \sum_{m=0}^{M} \sum_{n=0}^{N} B_{mn} y^{2m} z^n \tag{9}$$

$$Q_y(y,z) = \sum_{m=0}^{M}\sum_{n=0}^{N} C_{mn} y^{2m+1} z^n \tag{10}$$

where A_{mn}, B_{mn} and C_{mn} are undetermined coefficients. Applying these polynomials to the right-hand side of equations (3)–(5), we obtain

$$\begin{aligned} J(y,z) = {} & e^{-z} + \frac{1}{4}\sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn} \\ & \times \int_{-b}^{b}\int_{0}^{c} \frac{S_1(\tau)}{\tau} y'^{2m} z'^{n} \, dz' \, dy' \\ & + \frac{1}{4} a_1 \sum_{m=0}^{M}\sum_{n=0}^{N} B_{mn} \\ & \times \int_{-b}^{b}\int_{0}^{c} \frac{S_2(\tau)}{\tau^2}(z-z') y'^{2m} z'^{n} \, dz' \, dy' \\ & + \frac{1}{4} a_1 \sum_{m=0}^{M}\sum_{n=0}^{N} C_{mn} \\ & \times \int_{-b}^{b}\int_{0}^{c} \frac{S_2(\tau)}{\tau^2}(y-y') y'^{2m+1} z'^{n} \, dz' \, dy' \end{aligned} \tag{11}$$

$$\begin{aligned} Q_z(y,z) = {} & e^{-z} + \frac{1}{4}\sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn} \\ & \times \int_{-b}^{b}\int_{0}^{c} \frac{S_2(\tau)}{\tau^2}(z-z') y'^{2m} z'^{n} \, dz' \, dy' \\ & + \frac{1}{4} a_1 \sum_{m=0}^{M}\sum_{n=0}^{N} B_{mn} \\ & \times \int_{-b}^{b}\int_{0}^{c} \frac{S_3(\tau)}{\tau^3}(z-z')^2 y'^{2m} z'^{n} \, dz' \, dy' \\ & + \frac{1}{4} a_1 \sum_{m=0}^{M}\sum_{n=0}^{N} C_{mn} \\ & \times \int_{-b}^{b}\int_{0}^{c} \frac{S_3(\tau)}{\tau^3}(y-y')(z-z') y'^{2m+1} z'^{n} \, dz' \, dy' \end{aligned} \tag{12}$$

$$\begin{aligned} Q_z(y,z) = {} & \frac{1}{4}\sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn} \\ & \times \int_{-b}^{b}\int_{0}^{c} \frac{S_2(\tau)}{\tau^2}(y-y') y'^{2m} z'^{n} \, dz' \, dy' \\ & + \frac{1}{4} a_1 \sum_{m=0}^{M}\sum_{n=0}^{N} B_{mn} \\ & \times \int_{-b}^{b}\int_{0}^{c} \frac{S_3(\tau)}{\tau^3}(y-y')(z-z') y'^{2m} z'^{n} \, dz' \, dy' \\ & + \frac{1}{4} a_1 \sum_{m=0}^{M}\sum_{n=0}^{N} C_{mn} \\ & \times \int_{-b}^{b}\int_{0}^{c} \frac{S_3(\tau)}{\tau^3}(y-y')^2 y'^{2m+1} z'^{n} \, dz' \, dy'. \end{aligned} \tag{13}$$

In the collocation method, we force the left-hand side of equations (11)–(13) to be equal to the assumed polynomials for $J(y,z)$, $Q_z(y,z)$ and $Q_y(y,z)$, respectively, at $(M+1)(N+1)$ collocation points. In this work, we choose the Gaussian points to be the collocation points [9]. This generates $3(M+1)(N+1)$ algebraic equations for the determination of A_{mn}, B_{mn} and C_{mn}.

4. RESULTS AND DISCUSSION

Once A_{mn}, B_{mn} and C_{mn} are determined, there are two ways to compute $J(y,z)$, $Q_z(y,z)$ and $Q_y(y,z)$. One applies equations (8)–(10) and the other applies equations (11)–(13). Since the development of equations (11)–(13) is an iterative procedure, a small difference between the results using the two sets of equations reveals the accuracy of the results. This is illustrated in Table 1. As expected, when the difference between the results using equations (8)–(10) and those using equations (11)–(13) is small, the agreement of the present results with Crosbie and Schrenker's results [5] for isotropic scattering is excellent. Furthermore, the expansion with $M = 3$ and $N = 6$ is applied to an anisotropically scattering rectangular medium with a large aspect ratio, $2b/c = 8$. Due to the lack of results for two-dimensional anisotropic scattering, comparisons are made with one-dimensional geometry exposed to collimated radiation, as shown in Fig. 1. At the center of the medium the present solutions approach one-dimensional iteration solutions [4]. The consistency of the results for the limiting cases shows the validity of the present method.

Table 1. Comparison of the total radiation intensity at $y = 0$ for a variety of optical sizes ($a_1 = 0$)

b	c	z/c	Equations (8)–(10)	Equations (11)–(13)	Ref. [5]
0.125	0.25	0.0000	1.134	1.130	1.130
		0.1519	1.140	1.140	1.140
		0.3731	1.107	1.107	1.107
		0.5000	1.078	1.078	1.078
		0.6269	1.045	1.044	1.044
		0.8481	0.9720	0.9719	0.9718
		1.0000	0.9002	0.8972	0.8972
0.5	1.0	0.0000	1.436	1.416	1.416
		0.1519	1.475	1.474	1.474
		0.3731	1.340	1.340	1.340
		0.5000	1.229	1.229	1.229
		0.6269	1.106	1.105	1.105
		0.8481	0.8653	0.8650	0.8648
		1.0000	0.6555	0.6470	0.6472
2.0	4.0	0.0000	2.117	2.010	2.011
		0.1519	2.301	2.297	2.295
		0.3731	1.743	1.744	1.744
		0.5000	1.384	1.384	1.383
		0.6269	1.055	1.054	1.052
		0.8481	0.5627	0.5631	0.5627
		1.0000	0.2484	0.2411	0.2423

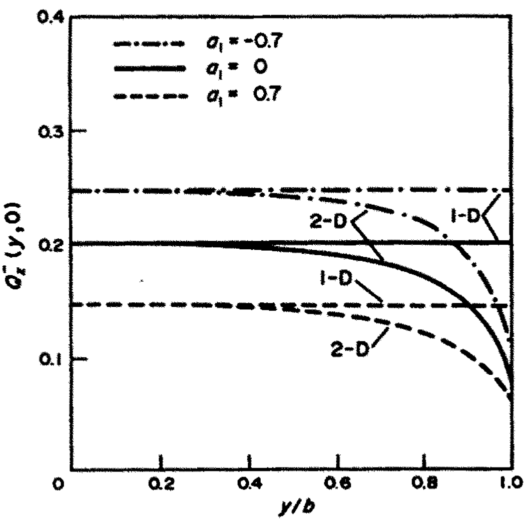

FIG. 1. Comparisons of one-dimensional ($c = 0.5$) and two-dimensional ($b = 2.0$, $c = 0.5$) anisotropic scattering: the flux leaving the bottom surface, $Q_z^-(y, 0)$.

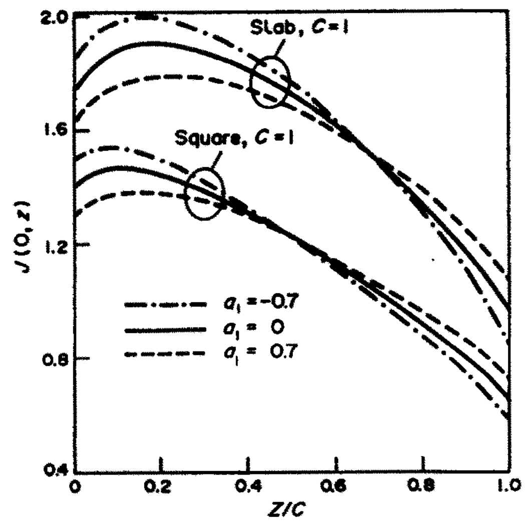

FIG. 2. The total radiation intensity at $y = 0$, $J(0, z)$, for a slab and a square.

Figure 1 shows that the outward flux decreases as the location considered moves to the corner. The corner effects are because part of the radiation is scattered out of the side surface before reaching the bottom. Backward scattering, say $a_1 = -0.7$, always generates the largest outward flux at the bottom, $Q_z^-(y, 0)$, while forward scattering, say $a_1 = 0.7$, generates the smallest, as shown in Fig. 1. This is because radiation originating at a point in the medium or at a boundary is more easily reflected back into the surroundings due to a relatively larger backward scattering. Besides, the incident radiation penetrating a medium decreases with optical size. This tendency is the same as that found in isotropic scattering [5] (see also Table 1).

Figure 2 shows that (i) the total intensity at the locations around the bottom is increased by backward scattering, but is decreased by forward scattering, (ii) the maximum of the total intensity locates about 10% of the optical thickness above the bottom, and (iii) the total intensity in a square medium is less than that in a slab with the same optical thickness.

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A linear stability analysis of a mixed convection plume

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INTRODUCTION

ANALYSES of laminar mixed convection from a horizontal line source of heat have been reported in a number of recent studies. These include the earliest by Wood [1], followed by those of Wesseling [2], Afzal [3] and Krishnamurthy and Gebhart [4]. All these studies were primarily concerned with the predictions of velocity and temperature fields.

In this paper, the stability of such flows to small disturbances is investigated in terms of the linear stability theory. The buoyancy force and the free stream flow are taken to be in the same direction. The region sufficiently downstream of the source is considered, where buoyancy effects dominate. This flow configuration, is usually termed aiding mixed convection.

The effect of the free stream is considered as a perturbation in the far-field boundary condition on the tangential velocity component of the natural convection plume. This perturbation is termed the mixed convection effect and is characterized by the parameter ε_M. Also taken into account is the first-order correction to the 'classical' boundary layer solution to the natural convection plume. This correction results from the interaction of the plume with the irrotational flow outside the boundary layer. This perturbation is termed the higher-order effect and is characterized by ε_H. The base flow is taken to be the classical natural convection plume perturbed by ε_M and ε_H. The stability analysis is then performed by expanding the disturbance field too, in terms of these two perturbation parameters. These two perturbation parameters have been so chosen that at zero order, the governing equations reduce to that of the laminar natural convection plume. Computed results are presented and discussed for $Pr = 0.7$.

ANALYSIS

The mixed convection flow arising from an infinitely long horizontal line source of heat is considered as a two-dimensional steady flow. With the usual Boussinesq approxi-